Pensieve Header: Poly-only β-Alexander.

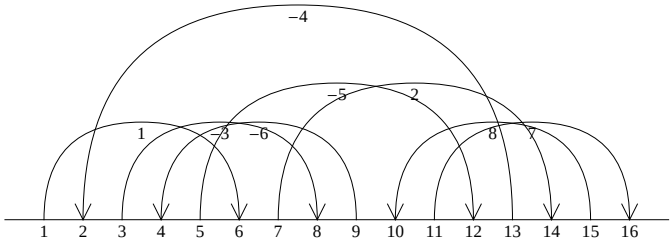
In[1]:=
```
<< KnotTheory`
GD[K_] := GD @@ (
    PD[K] /. X[i_, j_, k_, l_] :> If[PositiveQ[X[i, j, k, l]],
        Ar[l, i, +1], Ar[j, i, -1]
       ]
  )
```

```
Loading KnotTheory` version of August 22, 2010, 13:36:57.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[3]:=
```
Draw[expr_] := expr /. gd_GD :> Draw[gd];
Draw[gd_GD] := Module[
    {n = Length[gd], h, k = 0},
    Graphics[{
      Line[{{0, 0}, {2 n + 1, 0}}],
      Table[Text[i, {i, -0.3}], {i, 2 n}],
      (List @@ gd) /. {
        Ar[i_, j_, s_] :> {
          h = Abs[i - j] / 2;
          BezierCurve[{
            {i, 0}, {i, h}, {(i + j) / 2, h}, {j, h}, {j, 0}
           }, SplineDegree → 2],
          Text[s * (++k), {(i + j) / 2, h - 0.3}],
          Line[{{j - 0.2, 0.4}, {j, 0}, {j + 0.2, 0.4}}]
         }
       }
     }
    ]
  ];
Draw[GD[Knot[8, 17]]]
```

```
KnotTheory::loading : Loading precomputed data in PD4Knots`.
```

Out[5]=



In[6]:=
```
βSimplify = Factor;
SetAttributes[βCollect, Listable];
βCollect[B[ω_, σ_, μ_]] := B[
   βSimplify[ω],
   Collect[σ, _h, βSimplify],
   Collect[μ, _h, Collect[#, _t, βSimplify] &]
  ];
(* "L" for "Labels" *)
hL[β_] := Union[Cases[β, h[s_] :> s, Infinity]];
tL[β_] := Union[Cases[β, t[s_] | T_s_ :> s, Infinity]];
gL[β_] := Union[hL[β], tL[β]];
SetAttributes[βForm, Listable];
βForm[B[ω_, σ_, μ_]] := Module[
   {tails, heads, mat},
   tails = tL[B[ω, σ, μ]]; heads = hL[B[ω, σ, μ]];
   mat = Outer[βSimplify[Coefficient[μ, h[#1] t[#2]]] &, heads, tails];
   PrependTo[mat, t /@ tails];
   mat = Prepend[Transpose[mat], Prepend[h /@ heads, ω]];
   AppendTo[mat, Prepend[βSimplify[Coefficient[σ, h[#]]] & /@ heads, "σ"]];
   MatrixForm[mat]
  ];
```

In[14]:=
```
R[x_, y_] := B[1, Tx h[y], (Tx - 1) t[x] h[y]];
Rinv[x_, y_] := B[1, Tx^-1 h[y], (Tx^-1 - 1) t[x] h[y]];
tm[x_, y_, z_][β_] := β /. {t[x] → t[z], t[y] → t[z], Tx → Tz, Ty → Tz};
hm[x_, y_, z_][B[ω_, σ_, μ_]] := Module[
   {γx = D[μ, h[x]], γy = D[μ, h[y]], M = μ /. h[x] | h[y] → 0},
   B[ω,
     (σ /. h[x] | h[y] → 0) +
      h[z] Replace[D[σ, h[x]], 0 → 1] Replace[D[σ, h[y]], 0 → 1],
     M + h[z] (γx + Replace[D[σ, h[x]], 0 → 1] γy)
    ] // βCollect
  ];
swap[x_, y_][B[ω_, σ_, μ_]] := Module[
   {α, β, γ, δ, ϵ},
   α = Coefficient[μ, h[x] t[y]];
   β = D[μ, t[y]] /. h[x] → 0;
   γ = D[μ, h[x]] /. t[y] → 0;
   δ = μ /. h[x] | t[y] → 0;
   ϵ = ω / (ω + α);
   B[ω + α, σ, Plus[
      α / ϵ (1 + (Replace[D[σ, h[x]], 0 → 1] - 1 - α / ω) * ϵ) h[x] t[y],
      β / ϵ (1 + (Replace[D[σ, h[x]], 0 → 1] - 1 - α / ω) * ϵ) t[y],
      γ * h[x],
      (δ - ϵ * γ * β / ω) / ϵ
     ]] // βCollect
  ];
gm[x_, y_, z_][β_] := β // swap[y, x] // hm[x, y, z] // tm[x, y, z];
B /: B[ω1_, σ1_, μ1_] B[ω2_, σ2_, μ2_] := B[ω1 * ω2, σ1 + σ2, μ1 + μ2];
```

In[21]:=
```
{
 β = B[ω[T1, T2, T3, T4], 1, Sum[αi[T1, T2, T3, T4] t[i] h[1], {i, 4}]],
 β // tm[1, 2, 1],
 t1 = β // tm[1, 2, 1] // tm[1, 3, 1],
 t2 = β // tm[2, 3, 28] // tm[1, 28, 1],
 t1 == t2
} // βForm
```

Out[21]=
$$\left\{\begin{pmatrix} \omega[T_1, T_2, T_3, T_4] & h[1] \\ t[1] & \alpha_1[T_1, T_2, T_3, T_4] \\ t[2] & \alpha_2[T_1, T_2, T_3, T_4] \\ t[3] & \alpha_3[T_1, T_2, T_3, T_4] \\ t[4] & \alpha_4[T_1, T_2, T_3, T_4] \\ \sigma & 0 \end{pmatrix},\right.$$

$$\begin{pmatrix} \omega[T_1, T_1, T_3, T_4] & h[1] \\ t[1] & \alpha_1[T_1, T_1, T_3, T_4] + \alpha_2[T_1, T_1, T_3, T_4] \\ t[3] & \alpha_3[T_1, T_1, T_3, T_4] \\ t[4] & \alpha_4[T_1, T_1, T_3, T_4] \\ \sigma & 0 \end{pmatrix},$$

$$\begin{pmatrix} \omega[T_1, T_1, T_1, T_4] & h[1] \\ t[1] & \alpha_1[T_1, T_1, T_1, T_4] + \alpha_2[T_1, T_1, T_1, T_4] + \alpha_3[T_1, T_1, T_1, T_4] \\ t[4] & \alpha_4[T_1, T_1, T_1, T_4] \\ \sigma & 0 \end{pmatrix},$$

$$\left.\begin{pmatrix} \omega[T_1, T_1, T_1, T_4] & h[1] \\ t[1] & \alpha_1[T_1, T_1, T_1, T_4] + \alpha_2[T_1, T_1, T_1, T_4] + \alpha_3[T_1, T_1, T_1, T_4] \\ t[4] & \alpha_4[T_1, T_1, T_1, T_4] \\ \sigma & 0 \end{pmatrix}, \beta\text{Form}[\text{True}]\right\}$$

In[22]:=
```
{
 β = B[ω, 1, Sum[α10 i+j t[i] h[j], {i, 2}, {j, 4}]],
 β // hm[1, 2, 1],
 t1 = β // hm[1, 2, 1] // hm[1, 3, 1],
 t2 = β // hm[2, 3, 28] // hm[1, 28, 1],
 t1 == t2
} // βForm // ColumnForm
```

Out[22]=
$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] & h[4] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \sigma & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] & h[3] & h[4] \\ t[1] & \alpha_{11} + \alpha_{12} & \alpha_{13} & \alpha_{14} \\ t[2] & \alpha_{21} + \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \sigma & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] & h[4] \\ t[1] & \alpha_{11} + \alpha_{12} + \alpha_{13} & \alpha_{14} \\ t[2] & \alpha_{21} + \alpha_{22} + \alpha_{23} & \alpha_{24} \\ \sigma & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] & h[4] \\ t[1] & \alpha_{11} + \alpha_{12} + \alpha_{13} & \alpha_{14} \\ t[2] & \alpha_{21} + \alpha_{22} + \alpha_{23} & \alpha_{24} \\ \sigma & 1 & 0 \end{pmatrix}$$

βForm[True]

In[23]:=
```
Clear[β];
{β1 = B[ω, 1, h[1] t[1] α + h[2] t[1] β + h[1] t[2] γ + h[2] t[2] δ],
  β1 // swap[1, 1]
 } // βForm
```

Out[24]=
$$\left\{\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha & \beta \\ t[2] & \gamma & \delta \\ \sigma & 0 & 0 \end{pmatrix}, \begin{pmatrix} \alpha+\omega & h[1] & h[2] \\ t[1] & \alpha & \beta \\ t[2] & \gamma & \frac{-\beta\,\gamma+\alpha\,\delta+\delta\,\omega}{\omega} \\ \sigma & 0 & 0 \end{pmatrix}\right\}$$

In[25]:=
```
{
   β = B[ω, 1, Sum[α10 i+j t[i] h[j], {i, 2}, {j, 3}]],
   β // hm[1, 2, 1],
   t1 = β // hm[1, 2, 1] // swap[1, 1],
   t2 = β // swap[1, 1] // swap[2, 1] // hm[1, 2, 1],
   First[t1] == First[t2],
   Last[t1] == Last[t2] // Simplify
  } // βForm // ColumnForm
```

Out[25]=
$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \sigma & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] & h[3] \\ t[1] & \alpha_{11}+\alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21}+\alpha_{22} & \alpha_{23} \\ \sigma & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \omega+\alpha_{11}+\alpha_{12} & h[1] & h[3] \\ t[1] & \alpha_{11}+\alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21}+\alpha_{22} & \frac{-\alpha_{13}\,\alpha_{21}-\alpha_{13}\,\alpha_{22}+\omega\,\alpha_{23}+\alpha_{11}\,\alpha_{23}+\alpha_{12}\,\alpha_{23}}{\omega} \\ \sigma & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \omega+\alpha_{11}+\alpha_{12} & h[1] & h[3] \\ t[1] & \alpha_{11}+\alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21}+\alpha_{22} & \frac{-\alpha_{13}\,\alpha_{21}-\alpha_{13}\,\alpha_{22}+\omega\,\alpha_{23}+\alpha_{11}\,\alpha_{23}+\alpha_{12}\,\alpha_{23}}{\omega} \\ \sigma & 1 & 0 \end{pmatrix}$$

```
βForm[True]
βForm[True]
```

In[26]:=
```
{
  β = B[ω, 1, Sum[α10 i+j t[i] h[j], {i, 3}, {j, 2}]],
  t1 = β // tm[1, 2, 1] // swap[1, 1],
  t2 = β // swap[1, 2] // swap[1, 1] // tm[1, 2, 1],
  First[t1] == First[t2],
  Last[t1] == Last[t2] // Simplify
 } // βForm // ColumnForm
```

Out[26]=
$$\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha_{11} & \alpha_{12} \\ t[2] & \alpha_{21} & \alpha_{22} \\ t[3] & \alpha_{31} & \alpha_{32} \\ \sigma & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \omega+\alpha_{11}+\alpha_{21} & h[1] & h[2] \\ t[1] & \alpha_{11}+\alpha_{21} & \alpha_{12}+\alpha_{22} \\ t[3] & \alpha_{31} & \frac{-\alpha_{12}\,\alpha_{31}-\alpha_{22}\,\alpha_{31}+\omega\,\alpha_{32}+\alpha_{11}\,\alpha_{32}+\alpha_{21}\,\alpha_{32}}{\omega} \\ \sigma & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \omega+\alpha_{11}+\alpha_{21} & h[1] & h[2] \\ t[1] & \alpha_{11}+\alpha_{21} & \alpha_{12}+\alpha_{22} \\ t[3] & \alpha_{31} & \frac{-\alpha_{12}\,\alpha_{31}-\alpha_{22}\,\alpha_{31}+\omega\,\alpha_{32}+\alpha_{11}\,\alpha_{32}+\alpha_{21}\,\alpha_{32}}{\omega} \\ \sigma & 0 & 0 \end{pmatrix}$$

βForm[True]
βForm[True]

In[27]:=
```
{
  β = B[ω, 1, Sum[α10 i+j t[i] h[j], {i, 4}, {j, 4}]],
  t1 = β // gm[1, 2, 1] // gm[1, 3, 1],
  t2 = β // gm[2, 3, 2] // gm[1, 2, 1],
  First[t1] == First[t2],
  Last[t1] == Last[t2] // Simplify
 } // βForm // ColumnForm
```

Out[27]=
$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] & h[4] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ t[3] & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ t[4] & \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \\ \sigma & 0 & 0 & 0 & 0 \end{pmatrix}$$

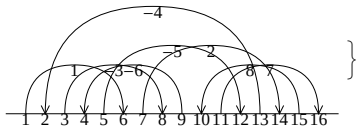
$$\begin{pmatrix} \frac{\omega^2+\omega\,\alpha_{12}+\omega\,\alpha_{13}-\alpha_{13}\,\alpha_{22}+\omega\,\alpha_{23}+\alpha_{12}\,\alpha_{23}}{\omega} & \frac{\omega^2\,\alpha_{11}+\omega^2\,\alpha_{12}+\omega^2\,\alpha_{13}+\omega^2\,\alpha_{21}+\omega\,\alpha_{12}\,\alpha_{21}+\omega^2\,\alpha_{22}-\omega\,\alpha_{11}\,\alpha_{22}-\omega\,\alpha_{13}\,\alpha_{22}+\omega^2\,\alpha_{23}+\omega\,\alpha_{12}\,\alpha_{23}+\omega^2\,\alpha_{31}+\omega\,\alpha_{12}\,\alpha_{31}+\omega\,\alpha_{13}\,\alpha_{31}}{\omega^2\,\alpha_{41}+\omega\,\alpha_{12}\,\alpha_{41}+\omega\,\alpha_{13}\,\alpha_{41}-\alpha_{13}\,\alpha_{22}\,\alpha_{41}+\omega\,\alpha_{23}\,\alpha_{41}+\alpha_{12}\,\alpha_{23}\,\alpha_{41}+\omega^2\,\alpha_{42}} \\ t[1] & \\ t[4] & \\ \sigma & \end{pmatrix}$$

$$\begin{pmatrix} \frac{\omega^2+\omega\,\alpha_{12}+\omega\,\alpha_{13}-\alpha_{13}\,\alpha_{22}+\omega\,\alpha_{23}+\alpha_{12}\,\alpha_{23}}{\omega} & \frac{\omega^2\,\alpha_{11}+\omega^2\,\alpha_{12}+\omega^2\,\alpha_{13}+\omega^2\,\alpha_{21}+\omega\,\alpha_{12}\,\alpha_{21}+\omega^2\,\alpha_{22}-\omega\,\alpha_{11}\,\alpha_{22}-\omega\,\alpha_{13}\,\alpha_{22}+\omega^2\,\alpha_{23}+\omega\,\alpha_{12}\,\alpha_{23}+\omega^2\,\alpha_{31}+\omega\,\alpha_{12}\,\alpha_{31}+\omega\,\alpha_{13}\,\alpha_{31}}{\omega^2\,\alpha_{41}+\omega\,\alpha_{12}\,\alpha_{41}+\omega\,\alpha_{13}\,\alpha_{41}-\alpha_{13}\,\alpha_{22}\,\alpha_{41}+\omega\,\alpha_{23}\,\alpha_{41}+\alpha_{12}\,\alpha_{23}\,\alpha_{41}+\omega^2\,\alpha_{42}} \\ t[1] & \\ t[4] & \\ \sigma & \end{pmatrix}$$

βForm[True]
βForm[True]

In[28]:=
```
{Alexander[K = Knot[8, 17]][X], Draw[GD[K]]}
```

Out[28]=
$$\left\{11 - \frac{1}{X^3} + \frac{4}{X^2} - \frac{8}{X} - 8\,X + 4\,X^2 - X^3,\right.$$



$$\left.\vphantom{\frac{1}{X}}\right\}$$

In[29]:=
```
(β = Times @@ GD[K] /. {Ar[x_, y_, +1] :> R[x, y], Ar[x_, y_, -1] :> Rinv[x, y]}) //
 βForm
```

Out[29]//MatrixForm=

	1	h[2]	h[4]	h[6]	h[8]	h[10]	h[12]	h[14]	h[16]
t[1]	0	0	$-1+T_1$	0	0	0	0	0	
t[3]	0	0	0	$-\frac{-1+T_3}{T_3}$	0	0	0	0	
t[5]	0	0	0	0	0	$-\frac{-1+T_5}{T_5}$	0	0	
t[7]	0	0	0	0	0	0	$-1+T_7$	0	
t[9]	0	$-\frac{-1+T_9}{T_9}$	0	0	0	0	0	0	
t[11]	0	0	0	0	0	0	0	$-1+T_{11}$	
t[13]	$-\frac{-1+T_{13}}{T_{13}}$	0	0	0	0	0	0	0	
t[15]	0	0	0	0	$-1+T_{15}$	0	0	0	
σ	$\frac{1}{T_{13}}$	$\frac{1}{T_9}$	T_1	$\frac{1}{T_3}$	T_{15}	$\frac{1}{T_5}$	T_7	T_{11}	

In[30]:=
```
β = Times @@ GD[K] /. {Ar[x_, y_, +1] :> R[x, y], Ar[x_, y_, -1] :> Rinv[x, y]};
Table[
  {k, (β = β // gm[1, k, 1]) // βForm, Collect[Last[β] /. t[i_] :> 1, _h, Simplify]},
  {k, 2, 2 Crossings[K]}
 ] // ColumnForm
```

Out[31]=

$\{2,$

	1	h[1]	h[4]	h[6]	h[8]	h[10]	h[12]	h[14]	h[16]
t[1]	0	0	$\frac{-1+T_1}{T_{13}}$	0	0	0	0	0	
t[3]	0	0	0	$-\frac{-1+T_3}{T_3}$	0	0	0	0	
t[5]	0	0	0	0	0	$-\frac{-1+T_5}{T_5}$	0	0	
t[7]	0	0	0	0	0	0	$-1+T_7$	0	
t[9]	0	$-\frac{-1+T_9}{T_9}$	0	0	0	0	0	0	
t[11]	0	0	0	0	0	0	0	$-1+T_{11}$	
t[13]	$-\frac{-1+T_{13}}{T_{13}}$	0	$\frac{(-1+T_1)(-1+T_{13})}{T_{13}}$	0	0	0	0	0	
t[15]	0	0	0	0	$-1+T_{15}$	0	0	0	
σ	$\frac{1}{T_{13}}$	$\frac{1}{T_9}$	T_1	$\frac{1}{T_3}$	T_{15}	$\frac{1}{T_5}$	T_7	T_{11}	

$, h[6]\,(-1+T_1)\}$

$\{3,$

	1	h[1]	h[4]	h[6]	h[8]	h[10]	h[12]	h[14]	h[16]
t[1]	0	0	$\frac{-1+T_1}{T_{13}}$	$-\frac{-1+T_1}{T_1}$	0	0	0	0	
t[5]	0	0	0	0	0	$-\frac{-1+T_5}{T_5}$	0	0	
t[7]	0	0	0	0	0	0	$-1+T_7$	0	
t[9]	0	$-\frac{-1+T_9}{T_9}$	0	0	0	0	0	0	
t[11]	0	0	0	0	0	0	0	$-1+T_{11}$	
t[13]	$-\frac{-1+T_{13}}{T_{13}}$	0	$\frac{(-1+T_1)(-1+T_{13})}{T_{13}}$	0	0	0	0	0	
t[15]	0	0	0	0	$-1+T_{15}$	0	0	0	
σ	$\frac{1}{T_{13}}$	$\frac{1}{T_9}$	T_1	$\frac{1}{T_1}$	T_{15}	$\frac{1}{T_5}$	T_7	T_{11}	

$, h[8]\left(-1+\frac{1}{T_1}\right)\}$

$$\left\{4,\begin{pmatrix} 1 & h[1] & h[6] & h[8] & h[10] & h[12] & h[14] & h[16] \\ t[1] & 0 & \frac{-1+T_1}{T_9\,T_{13}} & -\frac{-1+T_1}{T_1\,T_9} & 0 & 0 & 0 & 0 \\ t[5] & 0 & 0 & 0 & 0 & -\frac{-1+T_5}{T_5} & 0 & 0 \\ t[7] & 0 & 0 & 0 & 0 & 0 & -1+T_7 & 0 \\ t[9] & -\frac{-1+T_9}{T_9\,T_{13}} & \frac{(-1+T_1)\,(-1+T_9)}{T_9\,T_{13}} & -\frac{(-1+T_1)\,(-1+T_9)}{T_1\,T_9} & 0 & 0 & 0 & 0 \\ t[11] & 0 & 0 & 0 & 0 & 0 & 0 & -1+T_{11} \\ t[13] & -\frac{-1+T_{13}}{T_{13}} & \frac{(-1+T_1)\,(-1+T_{13})}{T_{13}} & 0 & 0 & 0 & 0 & 0 \\ t[15] & 0 & 0 & 0 & -1+T_{15} & 0 & 0 & 0 \\ \sigma & \frac{1}{T_9\,T_{13}} & T_1 & \frac{1}{T_1} & T_{15} & \frac{1}{T_5} & T_7 & T_{11} \end{pmatrix}, h[8]\left(-1+\frac{1}{T_1}\right)\right\}$$

$$\left\{5,\begin{pmatrix} 1 & h[1] & h[6] & h[8] & h[10] & h[12] & h[14] & h[16] \\ t[1] & 0 & \frac{-1+T_1}{T_9\,T_{13}} & -\frac{-1+T_1}{T_1\,T_9} & 0 & -\frac{-1+T_1}{T_1} & 0 & 0 \\ t[7] & 0 & 0 & 0 & 0 & 0 & -1+T_7 & 0 \\ t[9] & -\frac{-1+T_9}{T_9\,T_{13}} & \frac{(-1+T_1)\,(-1+T_9)}{T_9\,T_{13}} & -\frac{(-1+T_1)\,(-1+T_9)}{T_1\,T_9} & 0 & 0 & 0 & 0 \\ t[11] & 0 & 0 & 0 & 0 & 0 & 0 & -1+T_{11} \\ t[13] & -\frac{-1+T_{13}}{T_{13}} & \frac{(-1+T_1)\,(-1+T_{13})}{T_{13}} & 0 & 0 & 0 & 0 & 0 \\ t[15] & 0 & 0 & 0 & -1+T_{15} & 0 & 0 & 0 \\ \sigma & \frac{1}{T_9\,T_{13}} & T_1 & \frac{1}{T_1} & T_{15} & \frac{1}{T_1} & T_7 & T_{11} \end{pmatrix}, h[8]\left(-1+\frac{1}{T_1}\right)\right\}$$

$$\left\{6,\begin{pmatrix} \frac{-1+T_1+T_9\,T_{13}}{T_9\,T_{13}} & h[1] & h[8] & h[10] & h[12] & h[14] \\ t[1] & \frac{(-1+T_1)\,T_1}{T_9^2\,T_{13}^2} & -\frac{-1+T_1}{T_9} & 0 & 1-T_1 & 0 \\ t[7] & 0 & 0 & 0 & 0 & \frac{(-1+T_7)\,(-1+T_1+T_9\,T_{13})}{T_9\,T_{13}} \\ t[9] & -\frac{-1+T_9}{T_9\,T_{13}} & -\frac{(-1+T_1)\,(-1+T_9)}{T_1\,T_9} & 0 & \frac{(-1+T_1)^2\,(-1+T_9)}{T_1\,T_9\,T_{13}} & 0 \\ t[11] & 0 & 0 & 0 & 0 & 0 \\ t[13] & -\frac{-1+T_{13}}{T_{13}} & \frac{(-1+T_1)^2\,(-1+T_{13})}{T_1\,T_9\,T_{13}} & 0 & \frac{(-1+T_1)^2\,(-1+T_{13})}{T_1\,T_{13}} & 0 \\ t[15] & 0 & 0 & \frac{(-1+T_1+T_9\,T_{13})\,(-1+T_{15})}{T_9\,T_{13}} & 0 & 0 \\ \sigma & \frac{T_1}{T_9\,T_{13}} & \frac{1}{T_1} & T_{15} & \frac{1}{T_1} & T_7 \end{pmatrix}\right.$$

$$\left.\begin{matrix} \\ \\ \\ \\ \frac{(-1+T_{11}} \\ \\ \\ \end{matrix}\right.$$

$$\left\{7,\begin{pmatrix} \frac{-1+T_1+T_9\,T_{13}}{T_9\,T_{13}} & h[1] & h[8] & h[10] & h[12] & h[14] \\ t[1] & \frac{(-1+T_1)\,T_1}{T_9^2\,T_{13}^2} & -\frac{-1+T_1}{T_9} & 0 & 1-T_1 & \frac{(-1+T_1)\,(-1+T_1+T_9\,T_{13})}{T_9\,T_{13}} \\ t[9] & -\frac{-1+T_9}{T_9\,T_{13}} & -\frac{(-1+T_1)\,(-1+T_9)}{T_1\,T_9} & 0 & \frac{(-1+T_1)^2\,(-1+T_9)}{T_1\,T_9\,T_{13}} & 0 \\ t[11] & 0 & 0 & 0 & 0 & 0 \\ t[13] & -\frac{-1+T_{13}}{T_{13}} & \frac{(-1+T_1)^2\,(-1+T_{13})}{T_1\,T_9\,T_{13}} & 0 & \frac{(-1+T_1)^2\,(-1+T_{13})}{T_1\,T_{13}} & 0 \\ t[15] & 0 & 0 & \frac{(-1+T_1+T_9\,T_{13})\,(-1+T_{15})}{T_9\,T_{13}} & 0 & 0 \\ \sigma & \frac{T_1}{T_9\,T_{13}} & \frac{1}{T_1} & T_{15} & \frac{1}{T_1} & T_1 \end{pmatrix}\right.$$

$$\left.\begin{matrix} \\ \\ \\ \\ \frac{(-1+T_{11}} \\ \\ \\ \end{matrix}\right.$$

$$\left\{8,\ \begin{array}{|c|c|c|c|c|}
\frac{-1+T_1+T_{13}-T_1 T_{13}+T_9 T_{13}}{T_9 T_{13}} & h[1] & h[10] & h[12] & h[14 \\
t[1] & -\frac{(-1+T_1)(-1+T_{13})}{T_9^2 T_{13}^2} & 0 & -\frac{-1+T_1}{T_1} & \frac{(-1+T_1)(-1+ \\
t[9] & -\frac{-1+T_9}{T_9 T_{13}} & 0 & -\frac{(-1+T_1)^2(-1+T_9)(-1+T_{13})}{T_1 T_9 T_{13}} & \frac{(-1+T_1)^2(\\
t[11] & 0 & 0 & 0 & 0 \\
t[13] & \frac{(-1+T_1-T_9)(-1+T_{13})}{T_9 T_{13}} & 0 & \frac{(-1+T_1)^2(-1+T_{13})}{T_1 T_{13}} & -\frac{(-1+T_1)^3(\\
t[15] & 0 & \frac{(-1+T_1+T_{13}-T_1 T_{13}+T_9 T_{13})(-1+T_{15})}{T_9 T_{13}} & 0 & 0 \\
\sigma & \frac{1}{T_9 T_{13}} & T_{15} & \frac{1}{T_1} & T_1
\end{array}\right\}$$

$$\left\{9,\ \begin{array}{|c|c|c|c|c|}
\frac{-1+T_1+T_{13}}{T_1 T_{13}} & h[1] & h[10] & h[12] & h[14] \\
t[1] & -\frac{(-1+T_1)(-1+T_{13}+T_1 T_{13})}{T_1^2 T_{13}^2} & 0 & -\frac{(-1+T_1)\left(-1+2T_1-T_1^2+T_{13}-T_1 T_{13}+T_1^2 T_{13}\right)}{T_1^2 T_{13}} & \frac{(-1+T_1)\left(-1+T_1+T_{13}-T_1 \right.}{T_1^2 T_{13}} \\
t[11] & 0 & 0 & 0 & 0 \\
t[13] & -\frac{-1+T_{13}}{T_1 T_{13}} & 0 & \frac{(-1+T_1)^2(-1+T_{13})}{T_1 T_{13}} & -\frac{(-1+T_1)^3(-1+ \\
t[15] & 0 & \frac{(-1+T_1+T_{13})(-1+T_{15})}{T_1 T_{13}} & 0 & 0 \\
\sigma & \frac{1}{T_1 T_{13}} & T_{15} & \frac{1}{T_1} & T_1
\end{array}\right\}$$

$$\left\{10,\ \begin{array}{|c|c|c|c|}
\frac{-1+T_1+T_{13}}{T_1 T_{13}} & h[1] & h[12] & h[14] \\
t[1] & -\frac{(-1+T_1)(-1+T_{13}+T_1 T_{13})T_{15}}{T_1^2 T_{13}^2} & -\frac{(-1+T_1)\left(-1+2T_1-T_1^2+T_{13}-T_1 T_{13}+T_1^2 T_{13}\right)T_{15}}{T_1^2 T_{13}} & \frac{(-1+T_1)\left(-1+T_1+T_{13}-T_1 T_{13}+T_1^2 \right.}{T_1^2 T_{13}} \\
t[11] & 0 & 0 & 0 \\
t[13] & -\frac{-1+T_{13}}{T_1 T_{13}} & \frac{(-1+T_1)^2(-1+T_{13})}{T_1 T_{13}} & -\frac{(-1+T_1)^3(-1+T_{13})}{T_1^2 T_{13}} \\
t[15] & \frac{-1+T_{15}}{T_{13}} & \frac{(-1+T_1)\left(-1+2T_1-T_1^2+T_{13}-T_1 T_{13}+T_1^2 T_{13}\right)(-1+T_{15})}{T_1^2 T_{13}} & -\frac{(-1+T_1)\left(-1+T_1+T_{13}-T_1 T_{13}+T_1^2 T_{13} \right.}{T_1^2 T_{13}} \\
\sigma & \frac{T_{15}}{T_1 T_{13}} & \frac{1}{T_1} & T_1
\end{array}\right\}$$

$$\left\{11,\ \begin{array}{|c|c|c|c|}
\frac{-1+T_1+T_{13}}{T_1 T_{13}} & h[1] & h[12] & h[14] \\
t[1] & -\frac{(-1+T_1)(-1+T_{13}+T_1 T_{13})T_{15}}{T_1^2 T_{13}^2} & -\frac{(-1+T_1)\left(-1+2T_1-T_1^2+T_{13}-T_1 T_{13}+T_1^2 T_{13}\right)T_{15}}{T_1^2 T_{13}} & \frac{(-1+T_1)\left(-1+T_1+T_{13}-T_1 T_{13}+T_1^2 \right.}{T_1^2 T_{13}} \\
t[13] & -\frac{-1+T_{13}}{T_1 T_{13}} & \frac{(-1+T_1)^2(-1+T_{13})}{T_1 T_{13}} & -\frac{(-1+T_1)^3(-1+T_{13})}{T_1^2 T_{13}} \\
t[15] & \frac{-1+T_{15}}{T_{13}} & \frac{(-1+T_1)\left(-1+2T_1-T_1^2+T_{13}-T_1 T_{13}+T_1^2 T_{13}\right)(-1+T_{15})}{T_1^2 T_{13}} & -\frac{(-1+T_1)\left(-1+T_1+T_{13}-T_1 T_{13}+T_1^2 T_{13} \right.}{T_1^2 T_{13}} \\
\sigma & \frac{T_{15}}{T_1 T_{13}} & \frac{1}{T_1} & T_1
\end{array}\right\}$$

$$\left\{12,\ \begin{array}{|c|c|}
-\frac{T_1-T_1^2-T_1 T_{13}+T_{15}-3T_1 T_{15}+3T_1^2 T_{15}-T_1^3 T_{15}-T_{13} T_{15}+2T_1 T_{13} T_{15}-2T_1^2 T_{13} T_{15}+T_1^3 T_{13} T_{15}}{T_1^2 T_{13}} & h[1] \\
t[1] & -\frac{(-1+T_1)T_{15}\left(-T_1+T_1 T_{13}+T_1^2 T_{13}-T_{15}+2T_1 T_{15}- \right.}{T_1^4 T_{13}^2} \\
t[13] & \frac{(-1+T_{13})\left(-T_1-T_{15}+2T_1 T_{15} \right.}{T_1^2 T_{13}} \\
t[15] & \frac{-1+T_{15}}{T_{13}} \\
\sigma & \frac{T_{15}}{T_1^2 T_{13}}
\end{array}\right\}$$

$$\left\{13,\ \begin{array}{|c|c|}
-\frac{T_1-2T_1^2+T_{15}-4T_1 T_{15}+5T_1^2 T_{15}-3T_1^3 T_{15}+T_1^4 T_{15}}{T_1^3} & h[1] \\
t[1] & \frac{(-1+T_1)\left(-T_1^4+T_1 T_{15}-T_1^2 T_{15}-2T_1^3 T_{15}+2T_1^4 T_{15}-2T_1^5 T_{15}+T_1^6 T_{15}+T_{15}^2-3T_1 T_{15}^2+2T_1^2 T_{15}^2-T_1^3 \right.}{T_1^6} \\
t[15] & \frac{-1+T_{15}}{T_1} \\
\sigma & \frac{T_{15}}{T_1^3}
\end{array}\right\}$$

$$\left\{14,\begin{pmatrix} -\frac{1-3T_1+4T_1^2-4T_1^3+T_1^4-T_{15}+4T_1T_{15}-7T_1^2T_{15}+7T_1^3T_{15}-4T_1^4T_{15}+T_1^5T_{15}}{T_1^3} & h[1] \\ t[1] & \frac{(-1+T_1)\left(-T_1^4+T_{15}-2T_1T_{15}+2T_1^2T_{15}-3T_1^3T_{15}+2T_1^4T_{15}-2T_1^5T_{15}+T_1^6\right)}{T_1^5} \\ t[15] & -\frac{\left(1-3T_1+3T_1^2-3T_1^3+T_1^4\right)(-1} \\ \sigma & \frac{T_{15}}{T_1^2} \end{pmatrix}\right.$$

$$\left\{15,\begin{pmatrix} -\frac{1-4T_1+8T_1^2-11T_1^3+8T_1^4-4T_1^5+T_1^6}{T_1^3} & h[1] & h[16] \\ t[1] & \frac{(-1+T_1)\left(1-4T_1+8T_1^2-11T_1^3+8T_1^4-4T_1^5+T_1^6\right)}{T_1^4} & -\frac{(-1+T_1)\left(1-4T_1+8T_1^2-11T_1^3+8T_1^4-4T_1^5+T_1^6\right)}{T_1^3} \\ \sigma & \frac{1}{T_1} & T_1 \end{pmatrix}, h[\right.$$

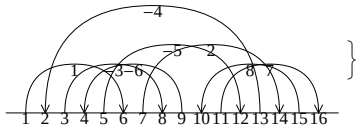
$$\left\{16,\begin{pmatrix} -\frac{1-4T_1+8T_1^2-11T_1^3+8T_1^4-4T_1^5+T_1^6}{T_1^2} & h[1] \\ t[1] & 0 \\ \sigma & 1 \end{pmatrix}, 0\right\}$$

In[32]:= β

Out[32]= $B\left[-\frac{1-4T_1+8T_1^2-11T_1^3+8T_1^4-4T_1^5+T_1^6}{T_1^2}, h[1], 0\right]$

In[33]:= **{Alexander[K = Knot[8, 17]][X], Draw[GD[K]]}**

Out[33]= $\left\{11-\frac{1}{X^3}+\frac{4}{X^2}-\frac{8}{X}-8X+4X^2-X^3,\right.$



$\left.\right\}$

In[34]:=
```
β = Times @@ GD[K] /. {Ar[x_, y_, +1] :> R[x, y], Ar[x_, y_, -1] :> Rinv[x, y]};
indices = Range[2 Crossings[K]];
Table[
  j = RandomInteger[{1, k - 1}];
  β = β // gm[indices[[j]], indices[[j + 1]], indices[[j]]];
  indices = Delete[indices, j + 1];
  {indices, β // βForm; Collect[Last[β] /. t[i_] :> 1, _h, Simplify[1 + #] &]},
  {k, 2 Crossings[K], 2, -1}
 ] // ColumnForm
```

Out[36]=

$$\left\{\{1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}, h[6]\, T_1 + \frac{h[8]}{T_3} + \frac{h[12]}{T_4} + h[14]\, T_7 + \frac{h[4]}{T_9} + h[16]\right.$$

$$\{\{1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}, h[6]\, T_1 + \frac{h[8]}{T_3} + \frac{h[12]}{T_4} + h[14]\, T_7 + \frac{h[4]}{T_9} + h[15]\, T_{11}$$

$$\{\{1, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}, h[6]\, T_1 + \frac{h[8]}{T_3} + \frac{h[12]}{T_4} + h[14]\, T_7 + \frac{h[4]}{T_9} + h[15]\, T_{11} + \frac{h}{}$$

$$\{\{1, 3, 4, 6, 7, 8, 10, 11, 12, 13, 14, 15\}, h[6]\, T_1 + \frac{h[8]}{T_3} + \frac{h[12]}{T_4} + h[14]\, T_7 + \frac{h[4]}{T_8} + h[15]\, T_{11} + \frac{h[1]}{T_{13}}$$

$$\{\{1, 4, 6, 7, 8, 10, 11, 12, 13, 14, 15\}, \frac{h[8]}{T_1} + h[6]\, T_1 + \frac{h[12]}{T_4} + h[14]\, T_7 + \frac{h[4]}{T_8} + h[15]\, T_{11} + \frac{h[1]}{T_{13}} + h$$

$$\{\{1, 4, 6, 7, 8, 10, 12, 13, 14, 15\}, \frac{h[8]}{T_1} + h[6]\, T_1 + \frac{h[12]}{T_4} + h[14]\, T_7 + \frac{h[4]}{T_8} + h[15]\, T_{10} + \frac{h[1]}{T_{13}} + h[10]$$

$$\{\{1, 6, 7, 8, 10, 12, 13, 14, 15\}, \frac{h[8]}{T_1} + \frac{h[12]}{T_1} + h[6]\, T_1 + h[14]\, T_7 + h[15]\, T_{10} + \frac{h[1]}{T_8\, T_{13}} + h[10]\, T_{15}\}$$

$$\{\{1, 6, 8, 10, 12, 13, 14, 15\}, \frac{h[8]}{T_1} + \frac{h[12]}{T_1} + h[6]\, T_1 + h[14]\, T_6 + h[15]\, T_{10} + \frac{h[1]}{T_8\, T_{13}} + h[10]\, T_{15}\}$$

$$\{\{1, 8, 10, 12, 13, 14, 15\}, \frac{h[8]\,(-1+2\,T_1-T_1^2+T_8\,T_{13})}{T_1\,T_8\,T_{13}} + \frac{h[12]\,(-1+2\,T_1-T_1^2+T_8\,T_{13})}{T_1\,T_8\,T_{13}} + \frac{h[1]\,(-T_1+T_1^2+T_8\,T_{13})}{T_8^2\,T_{13}^2} + h[14]\,\left(1 + \right.$$

$$\{\{1, 8, 10, 12, 13, 15\}, \frac{h[8]\,(-1+2\,T_1-T_1^2+T_8\,T_{13})}{T_1\,T_8\,T_{13}} + \frac{h[12]\,(-1+2\,T_1-T_1^2+T_8\,T_{13})}{T_1\,T_8\,T_{13}} + \frac{h[1]\,(-T_1+T_1^2+T_8\,T_{13})}{T_8^2\,T_{13}^2} + h[13]\,\left(1 + \frac{(-1+T}{}\right.$$

$$\{\{1, 8, 10, 12, 13\}, \frac{h[8]\,(-1+2\,T_1-T_1^2+T_8\,T_{13})}{T_1\,T_8\,T_{13}} + \frac{h[12]\,(-1+2\,T_1-T_1^2+T_8\,T_{13})}{T_1\,T_8\,T_{13}} + \frac{h[1]\,(-T_1+T_1^2+T_8\,T_{13})}{T_8^2\,T_{13}^2} + \frac{h[10]\,(1+T_1\,(-1+T_{13})-T_{13}+}{T_8\,T_{13}}$$

$$\{\{1, 8, 10, 12\}, \frac{h[8]\,(-1+2\,T_1-T_1^2+T_8\,T_{12})}{T_1\,T_8\,T_{12}} + \frac{h[1]\,(-T_1+T_1^2+T_8\,T_{12})}{T_8^2\,T_{12}^2} + \frac{h[10]\,(1+T_1\,(-1+T_{12})-T_{12}+T_8\,T_{12}^2)}{T_8\,T_{12}} + \frac{h[12]\,(1+T_1\,(-1+T_{10})+T_{10}}{T_8\,T_{12}}$$

$$\{\{1, 8, 12\}, \frac{h[1]\,(-T_1+T_1^2+T_8\,T_{12})}{T_8^2\,T_{12}^2} + \frac{h[12]\,(1+T_1\,(-1+T_8)-T_8+T_8^2\,T_{12})}{T_8\,T_{12}} + \frac{h[8]\,(-T_1^2+T_1\,(1+T_{12})+T_{12}\,(-1+T_8\,T_{12}))}{T_1\,T_8\,T_{12}}\}$$

$$\{\{1, 12\}, \frac{h[12]\,(1+3\,T_1\,(-1+T_{12})-4\,T_1^2\,(-1+T_{12})-T_1^4\,(-1+T_{12})-T_{12}+T_1^3\,(-3+4\,T_{12}))}{T_1^2\,T_{12}} + \frac{h[1]\,(-1-3\,T_1\,(-1+T_{12})-3\,T_1^3\,(-1+T_{12})+T_1^4\,(-1+T_{12})+T_1}{T_1^3\,T_{12}}$$

$$\{\{1\}, 1\}$$

In[37]:=
```
β // βForm
```

Out[37]//MatrixForm=

$$\begin{pmatrix} -\frac{1-4\,T_1+8\,T_1^2-11\,T_1^3+8\,T_1^4-4\,T_1^5+T_1^6}{T_1^2} & h[1] \\ t[1] & 0 \\ \sigma & 1 \end{pmatrix}$$